NANO-MICRO LETTERS

Vol. 2, No. 3
139-148 (2010)

State-space approach to vibration of gold nano-beam induced by ramp type heating

Hamdy M. Youssef[1,*] and Khaled A. Elsibai[2]

In the nanoscale beam, two effects become domineering. One is the non-Fourier effect in heat conduction and the other is the coupling effect between temperature and strain rate. In the present study, a generalized solution for the generalized thermoelastic vibration of gold nano-beam resonator induced by ramp type heating is developed. The solution takes into account the above two effects. State-space and Laplace transform methods are used to determine the lateral vibration, the temperature, the displacement, the stress and the strain energy of the beam. The effects of the relaxation time and the ramping time parameters have been studied.

Keywords: Thermoelasticity; Euler-Bernoulli equation; Gold nano-beam; Ramp type heating.

Citation: Hamdy M. Youssef and Khaled A. Elsibai, "State-Space approach to vibration of gold nano-beam induced by ramp type heating", Nano-Micro Lett. 2, 139-148 (2010). doi: 10.5101/nml.v2i3.p139-148

Many attempts have been made recently to investigate the elastic properties of nanostructured materials by atomistic simulations. Diao et al. [1] studied the effect of free surfaces on the structure and elastic properties of gold nanowires by atomistic simulations. Although the atomistic simulation is a good way to calculate the elastic constants of nanostructured materials, it is only applicable to homogeneous nanostructured materials (e.g., nanoplates, nanobeams, nanowires, etc.) with limited number of atoms. Moreover, it is difficult to obtain the elastic properties of the heterogeneous nanostructured materials using atomistic simulations. For these and other reasons, it is prudent to seek a more practical approach. One such approach would be to extend the classical theory of elasticity down to the nanoscale by including in it the hitherto neglected surface/interface effect. For this it is necessary first to cast the latter within the framework of continuum elasticity.

Nano-mechanical resonators have attracted considerable attention recently due to their many important technological applications. Accurate analysis of various effects on the characteristics of resonators, such as resonant frequencies and quality factors, is crucial for designing high-performance components. Many authors have studied the vibration and heat transfer process of beams. Kidawa [2] has studied the problem of transverse vibrations of a beam induced by a mobile heat source. The analytical solution to the problem was obtained using the Green's functions method. However, Kidawa did not consider the thermoelastic coupling effect. Boley [3] analyzed the vibrations of a simply supported rectangular beam subjected to a suddenly applied heat input distributed along its span. Manolis and Beskos [4] examined the thermally induced vibration of structures consisting of beams, exposed to rapid surface heating. They have also studied the effects of damping and axial loads on the structural response. Al-Huniti et al. [5] investigated the thermally induced displacements and stresses of a rod using the Laplace transformation technique. Ai Kah Soh et al. studied the vibration of micro/nanoscale beam resonators induced by ultra-short-pulsed laser by considering the thermoelastic coupling term in [6] and [7]. The propagation characteristics of the longitudinal wave in nanoplates with small scale effects are studied by Wang et al. [8].

When very fast phenomena and small structure dimensions are involved, the classical law of Fourier becomes inaccurate.

[1]Mathematical Department-Faculty of Education-Alexandria University-Egypt
[2]Mathematics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
*Corresponding author. E-mail: yousefanne@yahoo.com, drkhaledelsibai@yahoo.com

The classical Fourier heat conduction equation is a parabolic equation, whereas, the non-Fourier heat conduction equation is a hyperbolic equation. A more sophisticated model is then needed to describe the thermal conduction mechanisms in a physically acceptable way. Modern technology has enabled the fabrication of materials and devices with characteristic dimensions of a few nanometers. Examples are superlattices, nanowires, and quantum dots. At these length scales, the familiar continuum Fourier law for heat conduction is expected to fail due to both classical and quantum size effects [9]. Among many applications, the studying of the thermoelastic damping in MEMS /NEMS has been improved in [10] and [11].

It is worthwhile to mention here that in most of the earlier studies, mechanical or thermal loading on the bounding surface is considered to be in the form of a shock. However, the sudden jump of the load is merely an idealized situation because it is impossible to realize a pulse described mathematically by a step function; even very rapid rise-time (of the order of 10^{-9} s) may be slow in terms of the continuum. This is particularly true in the case of second sound effects when the thermal relaxation times for typical metals are less than 10^{-9} s Misra et al. [13]. It is thus felt that a finite time of rise of external load (mechanical or thermal) applied on the surface should be considered while studying a practical problem of this nature. Most ultrafast heat sources (such as certain lasers) involve the emission of a pulse (for example) that heats a material over a finite time due to the finite rise time of the pulse.

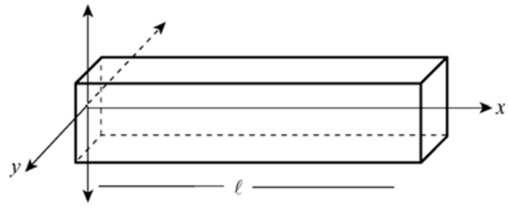
Considering the aspect of rise of time, Misra et al. [14] and Youssef with many authors investigated many applications in which the ramp-type heating is used [15-21].

State-space methods are the cornerstone of modern control theory. The essential feature of state-space methods is the characterization of the processes of interest by differential equations instead of transfer functions. This may seem like a throwback to the earlier, primitive period where differential equations also constituted the means of representing the behavior of dynamic processes. But in the earlier period, the processes were simple enough to be characterized by a single differential equation of fairly low order. In the modern approach, the processes are characterized by system of coupled, first-order differential equations. In principle, there is no limit to the order (i.e., the number of independent first-order differential equations), and in practice the only limit to the order is the availability of computer software capable of performing the required calculations reliably [22]. In particular, the state-space approach is useful because: (1) linear systems with time-varying parameters can be analyzed in essentially the same manner as time-invariant linear systems, (2) problems formulated by state-space methods can easily be programmed on a computer, (3) high-order linear systems can be analyzed, (4) multiple input-multiple output systems can be treated almost as easily as single input-single output linear systems, and (5) state-space theory is the foundation for further studies in such areas as nonlinear systems, stochastic systems, and optimal control. For solving coupled thermoelastic problems using the state-space approach in which the problem is rewritten in terms of state-space variables, namely, the temperature, the displacement and their gradients, has been developed by Bahar and Hetnarski [23-25].

In this paper, the non-Fourier effect in heat conduction, and the coupling effect between temperature and strain rate in nanoscale beam will be studied. In the present work, a generalized solution for the generalized thermoelastic vibration of gold nano-beam resonator induced by ramp type of heating will be developed. The state-space and the Laplace transform methods will be used to determine the lateral vibration, the temperature, the displacement, the stress and the strain energy of the beam. The effects of the relaxation time and the ramping time parameters will be studied and represented graphically.

Problem Formulation

Since beams with rectangular cross-sections are easy to fabricate, such cross-sections are commonly adopted in the design of NEMS resonators. Consider small flexural deflections of a thin elastic beam of length $\ell\left(0 \le x \le \ell\right)$, width $b\left(-\frac{b}{2} \le y \le \frac{b}{2}\right)$ and thickness $h\left(-\frac{h}{2} \le z \le \frac{h}{2}\right)$, for which the x, y and z axes are defined along the longitudinal, width and thickness directions of the beam, respectively. In equilibrium, the beam is unstrained, unstressed, and at temperature T_0 everywhere [6].



In the present study, the usual Euler-Bernoulli assumption [6,7] is adopted, i.e., any plane cross-section, initially perpendicular to the axis of the beam, remains plane and perpendicular to the neutral surface during bending. Thus, the displacements are given by

$$u=-z\frac{\partial w(x,t)}{\partial x}\ ,\ v=0,\ \ w(x,y,z,t)=w(x,t). \tag{1}$$

Hence, the differential equation of thermally induced lateral vibration of the beam may be expressed in the form [6]:

$$\frac{\partial^4 w}{\partial x^4}+\frac{\rho A}{EI}\frac{\partial^2 w}{\partial t^2}+\alpha_T\frac{\partial^2 M_T}{\partial x^2}=0, \tag{2}$$

where E is Young's modulus, I [= $bh^3/12$] the inertial moment about x-axis, ρ the density of the beam, α_T the coefficient of linear thermal expansion, $w(x,t)$ the lateral deflection, x the distance along the length of the beam, A=hb is the cross section area and t the time and M_T is the thermal moment, which is defined as:

$$M_T=\frac{12}{h^3}\int_{-h/2}^{h/2}\theta\, z\, dz, \tag{3}$$

where $\theta=T-T_0$ is the dynamical temperature increment of the resonator, in which T(x, z, t) is the temperature distribution and T_0 the environmental temperature.

The non-Fourier heat conduction equation has the following form [17]:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial z^2}=\left(\frac{\partial}{\partial t}+\tau_o\frac{\partial^2}{\partial t^2}\right)\left(\frac{\rho C_\upsilon}{k}\theta+\frac{\beta T_0}{k}e\right), \tag{4}$$

$e=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$ is the volumetric strain, C_υ is the specific heat at constant volume, τ_0 the thermal relaxation time, k the thermal conductivity, $\beta=\frac{E\alpha_T}{1-2\nu}$ in which ν is Poisson's ratio. Where there is no heat flow across the upper and lower surfaces of the beam, so that $\frac{\partial\theta}{\partial z}=0$ at $z=\pm h/2$ For a very thin beam and assuming the temperature varies in terms of a $\sin(pz)$ function along the thickness direction, where $p=\pi/h$, gives:

$$\theta(x,z,t)=\theta_1(x,t)\sin(pz).$$

Hence, equation (2) gives

$$\frac{\partial^4 w}{\partial x^4}+\frac{\rho A}{EI}\frac{\partial^2 w}{\partial t^2}+\frac{12\alpha_T}{h^3}\frac{\partial^2\theta_1}{\partial x^2}\int_{-h/2}^{h/2} z\sin(pz)dz=0 \tag{5}$$

and equation (4) gives

$$\frac{\partial^2\theta_1}{\partial x^2}\sin(pz)-p^2\theta_1\sin(pz)=\left(\frac{\partial}{\partial t}+\tau_o\frac{\partial^2}{\partial t^2}\right)\left(\frac{\rho C_\upsilon}{k}\theta_1\sin(pz)-\frac{\beta T_0}{k}z\frac{\partial^2 w}{\partial x^2}\right) \tag{6}$$

After doing the integrations, equation (5) takes the form

$$\frac{\partial^4 w}{\partial x^4}+\frac{\rho A}{EI}\frac{\partial^2 w}{\partial t^2}+\frac{24\alpha_T}{h\pi^2}\frac{\partial^2\theta_1}{\partial x^2}=0. \tag{7}$$

In equation (6), we multiply the both sides by z and integrating with respect to z from $-\frac{h}{2}$ to $\frac{h}{2}$ then we obtain

$$\left(\frac{\partial^2\theta_1}{\partial x^2}-p^2\theta_1\right)=\left(\frac{\partial}{\partial t}+\tau_o\frac{\partial^2}{\partial t^2}\right)\left(\eta\,\theta_1-\frac{\beta T_0\pi^2 h}{24k}\frac{\partial^2 w}{\partial x^2}\right), \tag{8}$$

Where, $\eta=\frac{\rho C_\upsilon}{k}$.

Now, for simplicity we will use the following non-dimensional variables:

$$(x',w',h')=\eta\, c_o(x,w,h)\ ,\ (t',\tau_o')=\eta\, c_o^2(t,\tau_o),$$
$$\sigma'=\frac{\sigma}{E},\ \theta_1'=\frac{\theta_1}{T_o},\ c_o^2=\frac{E}{\rho}. \tag{9}$$

Then, we have

$$\frac{\partial^4 w}{\partial x^4}+A_1\frac{\partial^2 w}{\partial t^2}+A_2\frac{\partial^2\theta_1}{\partial x^2}=0, \tag{10}$$

and

$$\frac{\partial^2\theta_1}{\partial x^2}-A_3\theta_1=\left(\frac{\partial}{\partial t}+\tau_o\frac{\partial^2}{\partial t^2}\right)\left(\theta_1-A_4\frac{\partial^2 w}{\partial x^2}\right), \tag{11}$$

Where

$$A_1=\frac{12}{h^2}\ ,\ A_2=\frac{24\alpha_t T_o}{\pi^2 h}\ ,\ A_3=p^2\ ,\ A_4=\frac{\pi^2\beta h}{24k\eta}\ ,$$

and we have canceled the prime for convenience.

Formulations the Problem in the Laplace Transform Domain

Applying the Laplace transform for equations (10) and (11) defined by the formula

$$\bar f(s)=\mathrm{L}[f(t)]=\int_0^\infty f(t)e^{-st}\,dt.$$

Hence, we obtain the following system of differential equations

$$\frac{d^4 \bar{w}}{d x^4} + A_1 s^2 \bar{w} + A_2 \frac{d^2 \bar{\theta}_1}{d x^2} = 0, \tag{12}$$

and

$$\frac{d^2 \bar{\theta}_1}{d x^2} - A_3 \bar{\theta}_1 = \left(s + \tau_o s^2\right)\left(\bar{\theta}_1 - A_4 \frac{d^2 \bar{w}}{d x^2}\right). \tag{13}$$

We will consider a new function as follows:

$$\frac{d^2 \bar{w}}{d x^2} = \bar{\eta}, \tag{14}$$

Then, we obtain

$$\frac{d^2 \bar{\theta}_1}{d x^2} = \alpha_1 \bar{\theta}_1 - \alpha_2 \bar{\eta}, \tag{15}$$

$$\frac{d^2 \bar{\eta}}{d x^2} = -\alpha_3 \bar{w} - \alpha_4 \bar{\theta}_1 + \alpha_5 \bar{\eta}, \tag{16}$$

where

$$\alpha_1 = \left(A_3 + s + \tau_o s^2\right), \qquad \alpha_2 = A_4\left(s + \tau_o s^2\right) \qquad \alpha_3 = A_1 s^2,$$

$$\alpha_4 = A_2\left(A_3 + s + \tau_o s^2\right), \ \alpha_5 = A_2 A_4\left(s + \tau_o s^2\right).$$

State-Space Formulation

Choosing as a state variable the functions $\bar{w}$, $\bar{\theta}_1$, $\bar{\eta}$, $\frac{d\bar{w}}{dx} = \bar{w}'$, $\frac{d\bar{\theta}_1}{dx} = \bar{\theta}_1'$ and $\frac{d\bar{\eta}}{dx} = \bar{\eta}'$ in the x-direction, then equations (14)-(16) can be written in matrix form by using the Bahar-Hetnarski method [23-25]:

$$\frac{d\bar{V}(x,s)}{d x} = A(s)\bar{V}(x,s) \tag{17}$$

Where

$$\bar{V}(x,s) = \begin{bmatrix} \bar{w}(x,s) \\ \bar{\theta}_1(x,s) \\ \bar{\eta}(x,s) \\ \bar{w}'(x,s) \\ \bar{\theta}_1'(x,s) \\ \bar{\eta}'(x,s) \end{bmatrix}, \tag{18}$$

and

$$A(s) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \alpha_1 & -\alpha_2 & 0 & 0 & 0 \\ -\alpha_3 & -\alpha_4 & \alpha_5 & 0 & 0 & 0 \end{bmatrix}. \tag{19}$$

The formal solution of equation (17) is given by

$$\bar{\mathrm{V}}(\mathrm{x},\mathrm{s}) = \exp[\mathrm{A}(\mathrm{s}).\mathrm{x}]\,\bar{\mathrm{V}}(0,\mathrm{s}), \tag{20}$$

Where

$$\bar{\mathbf{V}}(0,\mathrm{s}) = \begin{bmatrix} \bar{w}(0,s) \\ \bar{\theta}_1(0,s) \\ \bar{\eta}(0,s) \\ \bar{w}'(0,s) \\ \bar{\theta}_1'(0,s) \\ \bar{\eta}'(0,s) \end{bmatrix}. \tag{21}$$

The characteristic equation of the matrix A(s) has the form

$$\mathrm{k}^6 - \ell\, \mathrm{k}^4 + \mathrm{m}\, \mathrm{k}^2 - \mathrm{n} = 0, \tag{22}$$

Where

$$l = \alpha_1 + \alpha_5,\ m = \alpha_1\alpha_5 - \alpha_2\alpha_4 + \alpha_3,\ n = \alpha_1\alpha_3.$$

The roots of the characteristic equation (22) k_1^2, k_2^2 and k_3^2 satisfy the following relations

$$\mathrm{k}_1^2 + \mathrm{k}_2^2 + \mathrm{k}_3^2 = \ell, \tag{23a}$$

$$\mathrm{k}_1^2\mathrm{k}_2^2 + \mathrm{k}_2^2\mathrm{k}_3^2 + \mathrm{k}_1^2\mathrm{k}_3^2 = \mathrm{m}, \tag{23b}$$

$$\mathrm{k}_1^2\ \mathrm{k}_2^2\ \mathrm{k}_3^2 = \mathrm{n}. \tag{23c}$$

The Taylor series expansion for the matrix exponential is given by

$$\exp[\mathrm{A}(\mathrm{s})\mathrm{x}] = \sum_{\mathrm{i}=0}^{\infty} \frac{[\mathrm{A}(\mathrm{s})\ \mathrm{x}]^{\mathrm{i}}}{\mathrm{i}!}. \tag{24}$$

Using the Cayley-Hamilton theorem [23]-[25], this infinite series can be truncated to

$$\begin{aligned} &\exp[\mathrm{A}(\mathrm{s}).\mathrm{x}] = \mathrm{L}(\mathrm{x},\mathrm{s}) = \\ &\mathrm{a}_0\ \mathrm{I} + \mathrm{a}_1\ \mathrm{A} + \mathrm{a}_2\ \mathrm{A}^2 + \mathrm{a}_3\ \mathrm{A}^3 + \mathrm{a}_4\ \mathrm{A}^4 + \mathrm{a}_5\ \mathrm{A}^5, \end{aligned} \tag{25}$$

Where I is the unit matrix of order 6 and a_0 - a_5 are some parameters depending on s and x to be determined.

Using the Cayley-Hamilton theorem again [23,25], we obtain,

$$\begin{aligned} &\exp(k_1\mathrm{x}) = a_0 + a_1 k_1 + a_2 \mathrm{k}_1^2 + a_3 \mathrm{k}_1^3 + a_4 \mathrm{k}_1^4 + a_5 \mathrm{k}_1^5, \\ &\exp(-k_1\mathrm{x}) = a_0 - a_1 k_1 + a_2 \mathrm{k}_1^2 - a_3 \mathrm{k}_1^3 + a_4 \mathrm{k}_1^4 - a_5 \mathrm{k}_1^5, \\ &\exp(k_2\mathrm{x}) = a_0 + a_1 k_2 + a_2 \mathrm{k}_2^2 + a_3 \mathrm{k}_2^3 + a_4 \mathrm{k}_2^4 + a_5 \mathrm{k}_2^5, \\ &\exp(-\mathrm{k}_3\mathrm{x}) = \mathrm{a}_0 - \mathrm{a}_1\mathrm{k}_3 + \mathrm{a}_2\mathrm{k}_3^2 - \mathrm{a}_3\mathrm{k}_3^3 + \mathrm{a}_4\mathrm{k}_3^4 - \mathrm{a}_5\mathrm{k}_3^5, \\ &\exp(k_3\mathrm{x}) = a_0 + a_1 k_3 + a_2 \mathrm{k}_3^2 + a_3 \mathrm{k}_3^3 + a_4 \mathrm{k}_3^4 + a_5 \mathrm{k}_3^5, \\ &\exp(-\mathrm{k}_3\mathrm{x}) = \mathrm{a}_0 - \mathrm{a}_1\mathrm{k}_3 + \mathrm{a}_2\mathrm{k}_3^2 - \mathrm{a}_3\mathrm{k}_3^3 + \mathrm{a}_4\mathrm{k}_3^4 - \mathrm{a}_5\mathrm{k}_3^5. \end{aligned} \tag{26}$$

The solution of this system of linear equations is given by

$$\mathrm{a}_0 = -\mathrm{F}(\mathrm{k}_2^2\mathrm{k}_3^2\mathrm{c}_1 + \mathrm{k}_1^2\mathrm{k}_3^2\mathrm{c}_2 + \mathrm{k}_1^2\mathrm{k}_2^2\mathrm{c}_3),$$

$$\mathrm{a}_1 = -\mathrm{F}(\mathrm{k}_2^2\mathrm{k}_3^2\mathrm{s}_1 + \mathrm{k}_1^2\mathrm{k}_3^2\mathrm{s}_2 + \mathrm{k}_1^2\mathrm{k}_2^2\mathrm{s}_3),$$

$$
\begin{aligned}
a_2 &= F[(k_2^2+k_3^2)c_1+(k_3^2+k_1^2)c_2+(k_1^2+k_2^2)c_3],\\
a_3 &= F[(k_2^2+k_3^2)s_1+(k_3^2+k_1^2)s_2+(k_1^2+k_2^2)s_3],\\
a_4 &= -F(c_1+c_2+c_3),\\
a_5 &= -F(s_1+s_2+s_3).
\end{aligned} \tag{27}
$$

where

$$F=\frac{1}{(k_1^2-k_2^2)(k_2^2-k_3^2)(k_3^2-k_1^2)},$$

$$c_1=(k_2^2-k_3^2)\cosh(k_1 x),$$

$$c_2=(k_3^2-k_1^2)\cosh(k_2 x),$$

$$c_3(x,s)=(k_1^2-k_2^2)\cosh(k_3 x),$$

$$s_1(x,s)=\frac{(k_2^2-k_3^2)}{k_1}\sinh(k_1 x),$$

$$s_2(x,s)=\frac{(k_3^2-k_1^2)}{k_2}\sinh(k_2 x),$$

$$s_3(x,s)=\frac{(k_1^2-k_2^2)}{k_3}\sinh(k_3 x),$$

Substituting from equations (27) into equation (25), we obtain the matrix exponential in the form

$$\exp[A(s).x]=L(x,s)=[L_{ij}(x,s)],\quad i,j=1,2,3,4,5,6, \tag{28}$$

Where $L_{ij}(x,s)$, $i,j=1,2,3,4,5,6$ are defined in the appendix.

Now, we will consider the first end of the nano-beams x=0 is clamped and loaded thermally by ramp-type heating, which gives [6, 7]:

$$w(0,t)=\eta(0,t)=0, \tag{29}$$

and

$$\theta_1(0,t)=\theta_0\begin{bmatrix} 0 & \text{for } t\le 0\\ \dfrac{t}{t_0} & \text{for } 0<t<t_0\\ 1 & \text{for } t\ge t_0\end{bmatrix}, \tag{30}$$

Where t_0 is non-negative constant and is called ramp-type parameter and θ_0 is constant [19].

After using Laplace transform, the above conditions take the forms

$$\bar{w}(0,s)=\bar{\eta}(0,s)=0, \tag{31}$$

and

$$\bar{\theta}_1(0,s)=\frac{\theta_0}{t_0}\left(\frac{1-e^{-t_0 s}}{s^2}\right)=G(s). \tag{32}$$

Applying the conditions (31) and (32) into equations (21), we obtain

$$\bar{\mathbf{V}}(0,s)=\begin{bmatrix}0\\ G(s)\\ 0\\ \bar{w}'(0,s)\\ \bar{\theta}_{\mathrm{II}}'(0,s)\\ \bar{\eta}'(0,s)\end{bmatrix}. \tag{33}$$

To get $\bar{w}'(0,s)$, $\bar{\theta}_1'(0,s)$ and $\bar{\eta}'(0,s)$, we will consider the other end of the beam $x=\ell$ is clamped and remains at zero increment of temperature as follows:

$$w(\ell,t)=\theta_1(\ell,t)=\eta(\ell,t)=0. \tag{34}$$

After using Laplace transform, we have

$$\bar{w}(\ell,s)=\bar{\theta}_1(\ell,s)=\bar{\eta}(\ell,s)=0. \tag{35}$$

Hence, we obtain

$$\begin{bmatrix}\bar{w}'(0,s)\\ \bar{\theta}_{\mathrm{II}}'(0,s)\\ \bar{\eta}'(0,s)\end{bmatrix}=-G(s)\left(\begin{bmatrix}L_{14}(\ell,s) & L_{15}(\ell,s) & L_{16}(\ell,s)\\ L_{24}(\ell,s) & L_{25}(\ell,s) & L_{26}(\ell,s)\\ L_{34}(\ell,s) & L_{35}(\ell,s) & L_{36}(\ell,s)\end{bmatrix}^{-1}\begin{bmatrix}L_{12}(\ell,s)\\ L_{22}(\ell,s)\\ L_{32}(\ell,s)\end{bmatrix}\right). \tag{36}$$

After some complicated simplifications by using MAPLE software, we get the final solutions in the Laplace transform domain as follow:

The lateral deflection

$$\begin{aligned}\bar{w}(x,s)=&\frac{\Delta\sinh\big(k_1(\ell-x)\big)}{\big(k_1^2-k_2^2\big)\big(k_1^2-k_3^2\big)\sinh(k_1\ell)}\\ &+\frac{\Delta\sinh\big(k_2(\ell-x)\big)}{\big(k_2^2-k_1^2\big)\big(k_2^2-k_3^2\big)\sinh(k_2\ell)}\\ &+\frac{\Delta\sinh\big(k_3(\ell-x)\big)}{\big(k_3^2-k_1^2\big)\big(k_3^2-k_2^2\big)\sinh(k_3\ell)}.\end{aligned} \tag{37}$$

The temperature

$$\begin{aligned}\bar{\theta}(z,x,s)=&-\frac{\alpha_2 k_1^2\Delta\sin(pz)\sinh\big(k_1(\ell-x)\big)}{\big(k_1^2-\alpha_1\big)\big(k_1^2-k_2^2\big)\big(k_1^2-k_3^2\big)\sinh(k_1\ell)}\\ &-\frac{\alpha_2 k_2^2\Delta\sin(pz)\sinh\big(k_2(\ell-x)\big)}{\big(k_2^2-\alpha_1\big)\big(k_2^2-k_1^2\big)\big(k_2^2-k_1^2\big)\sinh(k_2\ell)}\\ &-\frac{\alpha_3 k_2^2\Delta\sin(pz)\sinh\big(k_3(\ell-x)\big)}{\big(k_3^2-\alpha_1\big)\big(k_3^2-k_1^2\big)\big(k_3^2-k_2^2\big)\sinh(k_3\ell)}.\end{aligned} \tag{38}$$

The displacement

$$\bar{u}(z,x,s) = -\frac{z\,\Delta\,k_1 \cosh\left(k_1\left(\ell - x\right)\right)}{\left(k_1^2 - k_2^2\right)\left(k_1^2 - k_3^2\right)\sinh\left(k_1\ell\right)} - \frac{z\,\Delta\,k_2 \cosh\left(k_2\left(\ell - x\right)\right)}{\left(k_2^2 - k_1^2\right)\left(k_2^2 - k_1^2\right)\sinh\left(k_2\ell\right)} - \frac{z\,\Delta\,k_3 \cosh\left(k_3\left(\ell - x\right)\right)}{\left(k_3^2 - k_1^2\right)\left(k_3^2 - k_2^2\right)\sinh\left(k_3\ell\right)}. \tag{39}$$

The Strain

$$\bar{e}(z,x,s) = \frac{z\,\Delta k_1^2 \sinh\left(k_1\left(\ell - x\right)\right)}{\left(k_1^2 - k_2^2\right)\left(k_1^2 - k_3^2\right)\sinh\left(k_1\ell\right)} + \frac{z\,\Delta\,k_2^2 \sinh\left(k_2\left(\ell - x\right)\right)}{\left(k_2^2 - k_1^2\right)\left(k_2^2 - k_1^2\right)\sinh\left(k_2\ell\right)} + \frac{z\,\Delta\,k_3^2 \sinh\left(k_3\left(\ell - x\right)\right)}{\left(k_3^2 - k_1^2\right)\left(k_3^2 - k_2^2\right)\sinh\left(k_3\ell\right)}. \tag{40}$$

Where

$$\Delta = \frac{G}{\alpha_1\alpha_2}\left(\alpha_1 - k_1^2\right)\left(\alpha_1 - k_2^2\right)\left(\alpha_1 - k_3^2\right)$$

The Stress and the Strain-Energy

The stress on the x-axis, according to Hooke's law is:

$$\sigma_{xx}\left(x,z,t\right) = E\left(e - \alpha_T\theta\right). \tag{41}$$

By using the non-dimensional variables in (9), we obtain the stress in the form

$$\sigma_{xx}\left(x,z,t\right) = e - \alpha_T T_0 \theta. \tag{42}$$

After using Laplace transform, the above equation takes the form:

$$\bar{\sigma}_{xx}\left(x,z,s\right) = \bar{e} - \alpha_T T_0 \bar{\theta}. \tag{43}$$

The strain energy which is generated on the beam is given by

$$W\left(x,z,t\right) = \sum_{i,j=1}^{3}\frac{1}{2}\sigma_{ij}e_{ij} = \frac{1}{2}\sigma_{xx}e_{xx} = -\frac{1}{2}z\sigma_{xx}\eta, \tag{44}$$

or, we can write as follows:

$$W\left(x,z,t\right) = -\frac{1}{2}z\left[L^{-1}\left(\bar{\sigma}_{xx}\right)\right]\left[L^{-1}\left(\bar{\eta}\right)\right], \tag{45}$$

Where $L^{-1}\left[\bar{f}\left(s\right)\right] = f\left(t\right)$ is the Laplace inverse.

Those complete the solution in the Laplace transform domain.

Numerical Inversion of the Laplace Transform

In order to determine the solutions in the time domain, the Riemann-sum approximation method is used to obtain the numerical results. In this method, any function in Laplace domain can be inverted to the time domain as

$$f(t) = \frac{e^{\kappa t}}{t}\left[\frac{1}{2}\bar{f}\left(\kappa\right) + \mathrm{Re}\sum_{n=1}^{N}\left(-1\right)^n \bar{f}\left(\kappa + \frac{in\pi}{t}\right)\right] \tag{46}$$

Where Re is the real part and i is imaginary number unit. For faster convergence, numerous numerical experiments have shown that the value of κ satisfies the relation $\kappa t \approx 4.7$ Tzou [26].

Numerical Results and Discussion

Now, we will consider a numerical example for which computational results are given. For this purpose, gold (Au) is taken as the thermoelastic material for which we take the following values of the different physical constants:

$k = 318\ W/\left(mK\right)$, $\alpha_T = 14.2\left(10\right)^{-6}\ \mathrm{K}^{-1}$,
$\rho = 1930\ \mathrm{kg/m^3}$, $T_0 = 293\mathrm{K}$, $C_\upsilon = 130\ J/\left(\mathrm{kg\,K}\right)$,
$E = 180\ \mathrm{GPa}$, $\upsilon = 0.44$.

The aspect ratios of the beam are fixed as $\ell/h = 10$ and $b/h = 1/2$. When h is varied, ℓ and b changed accordingly with h.

For the nanoscale beam, we will take the range of the beam length ℓ $\left(1-100\right)\times10^{-12}\,\mathrm{m}$. The original time t and the ramping time parameter t_0 will be considered in the picoseconds $\left(1-100\right)\times10^{-12}$ sec and the relaxation time τ_0 in the range $\left(1-100\right)\times10^{-14}$ sec.

The figures were prepared by using the non-dimensional variables which are defined in (9) for a wide range of beam length when $\ell = 1.0$, $\theta_0 = 1.0$ $z = h/6$ and $t = 0.15$.

Figures (1-5), represent the lateral vibration, the temperature, the displacement, the stress and the strain energy of the beam at different values of the relaxation time when $\tau_0 = 0.0$ (Biot) and $\tau_0 = 0.02$ (L-S) and we found that, the relaxation time has significant effects on all the studied fields. In the context of L-S, the values of the lateral vibration, the temperature, the displacement, the stress and the strain energy decreasing when the relaxation time value increases and it is very obvious in the peek points. In the context of L-S model, the speed of the wave propagation of all the studied fields vanish at points closed to the first edge of the beam more than the points at the context of Biot model and the damping of the strain-energy appear in L-S model before Biot model.

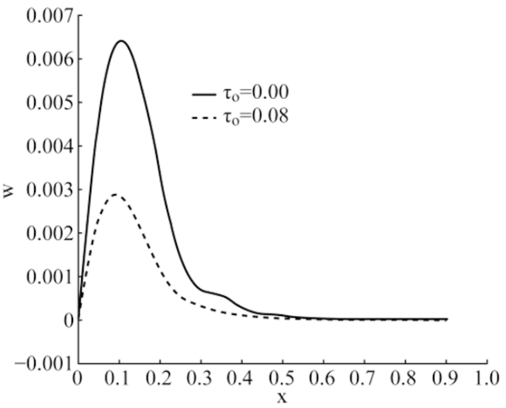


FIG. 1. The lateral deflection w For L-S and Biot theories

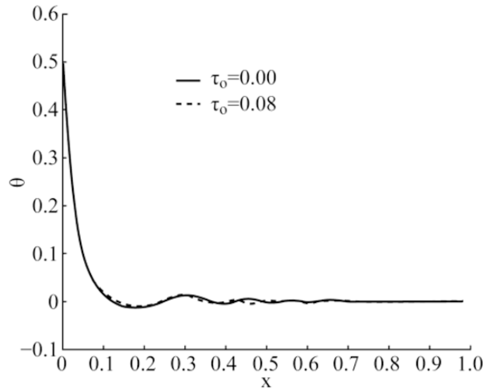


FIG. 2. The temperatue for L-S and Biot theories

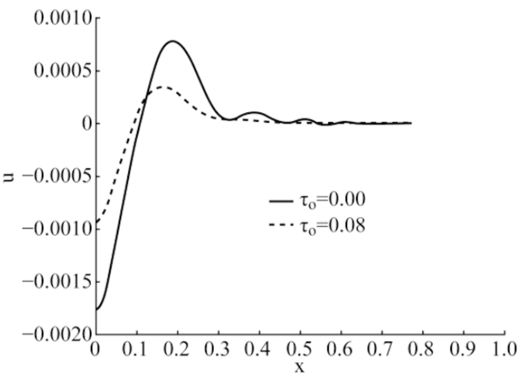


FIG. 3. The displacement for L-S and Biot theories

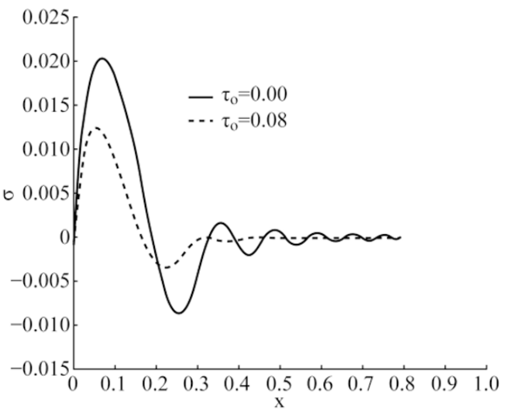


FIG. 4. The stress for L-S and Biot theories

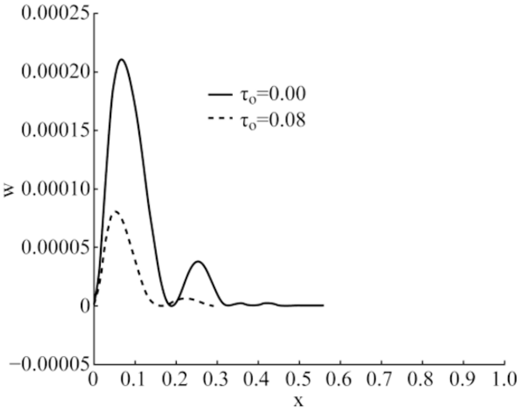


FIG. 5. The strain-energy at L-S and Biot theories

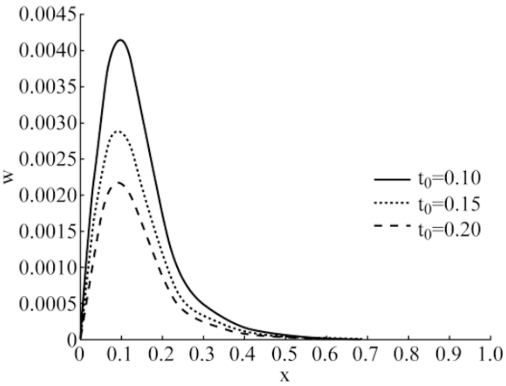


FIG. 6. The lateral deflection w at different time of ramping parameter

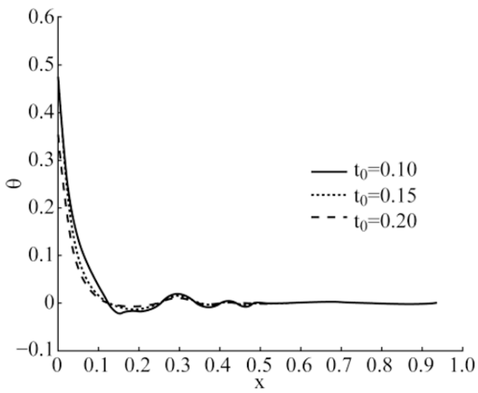


FIG. 7. The temperature at different time of ramping parameter

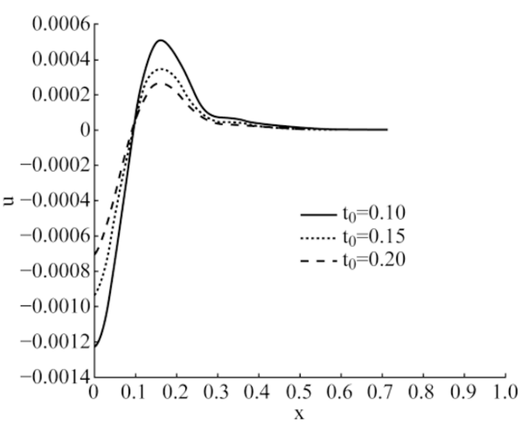


FIG. 8. The displacement at different time of ramping parameter

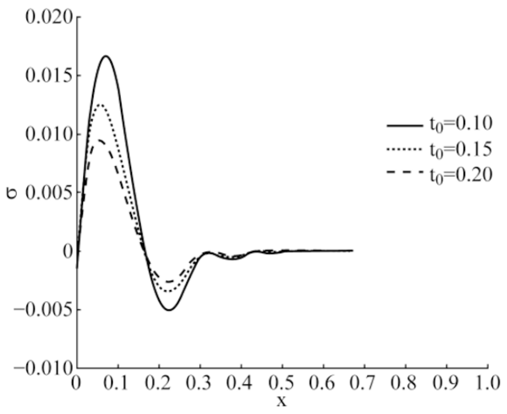


FIG. 9. The stress at different time of ramping parameter

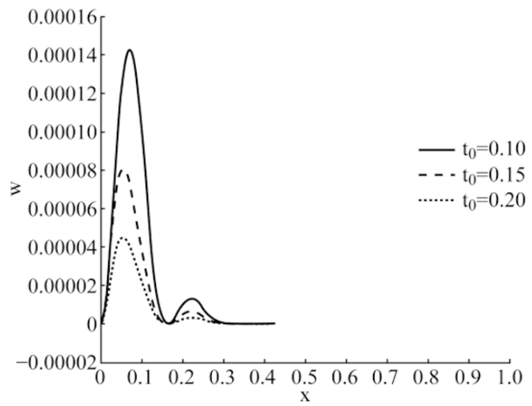


FIG. 10. The strain-energy at different value of ramping parameter

In figures (6-10), we represented the lateral vibration, the temperature, the displacement, the stress and the strain energy of the beam at different values of the ramping time parameter when $t_0(0.10)<t(0.15)$, $t_0= t=0.15$ and $t_0(0.20)>t(0.15)$ in the context of L-S model. We found that, the ramping time parameter has significant effects on all the studied fields. The increasing in the value of the ramping time parameter causes decreasing in the values of all the fields which is very obvious in the peek points of the curves. Also, the damping of the strain energy increases when the ramping time parameter increases.

Conclusion

This paper has investigated the vibration characteristics of the deflection, the temperature, the displacement, the stress and the strain energy of an Euler-Bernoulli gold nano-beam induced by a ramp type heating. An analytical direct method and numerical technique based on the Laplace transformation has been used to calculate the vibration of the deflection, the temperature, the displacement, the stress and the strain energy. The effects of the relaxation time and the ramping time parameter on all the studied fields have been shown and represented graphically. The non-Fourier law of heat conduction gives a finite speed of wave propagation and increases the damping of the strain energy.

Received 23 June 2010; accepted 12 July 2010; published online 26 July 2010.

References

1. J. K. Diao, K. Gall and M. L. Dunn, J. Mech. Phys. Solids 52, 1935 (2004). doi:10.1016/j.jmps.2004.03.009.
2. J. Kidawa-Kukla, J. Sound Vib. 262, 865 (2003).
3. B. A. Boley, J. Appl. Mech. 39, 212 (1972).
4. G. D. Manolis and D. E. Beskos, Comp. Methods in Appl. Mech. Engin. 21, 337 (1980).
5. N. S. Al-Huniti, M. A. Al-Nimr and M. Naij, J. Sound Vib. 242, 629 (2001).
6. Ai Kah Soh, Yuxin Sun and Daining Fang, J. Sound Vib. 311, 243 (2008). doi:10.1016/j.jsv.2007.09.002.
7. Yuxin Sun, Daining Fang, Masumi Saka and Ai Kah Soh, Inter. J. Solids Struct. 45, 1993 (2008). doi:10.1016/j.ijsolstr.2007.11.006.
8. Y. Z. Wang, F. M. Li and K. Kishimoto, Physica E 42, 1356 (2010). doi:10.1016/j.physe.2009.11.036.
9. J. S. Rao, Advanced theory of vibration (Nonlinear Vibration and One Dimensional Structures), Wiley, New York, Chichester, Brisbane, Toronto, Singapore, 1992.
10. Y. X. Sun, D. N. Fang and A. K. Soh, Inter. J. Solids Structures 43, 3213 (2006). doi:10.1016/j.ijsolstr.2005.08.011.
11. D. N. Fang, Y. X. Sun and A. K. Soh, Chin. Phys. Lett. 23, 1554 (2006) . doi:10.1088/0256-307X/23/6/055.
12. A. Duwel, J. Gorman, M. Weinstein, J. Borenstein and P. Ward, Sensor. Actuator. A 103, 70 (2003). doi:10.1016/S0924-4247(02)00318-7.
13. J. C. Misra, S. C. Samanta, A. K. Chakrabarti and S. C. Misra, Inter. J. Engin. Sci. 29, 1505 (1991). doi:10.1016/0020-7225(91)90122-J.
14. S. C. Misra, S. C. Samanta and A. K. Chakrabarti, Inter. J. Engin. Sci. 29, 1065 (1991). doi:10.1016/0020-7225(91)90112-G.
15. Hamdy M. Youssef and J. CAMQ, Appl. Math. Institute 13, 369 (2005).
16. Hamdy M. Youssef, Arch. Appl. Mech. 75, 553 (2006). doi:10.1007/s00419-005-0440-3.
17. Hamdy M. Youssef, Eur. J. Mech. A-Solid 25, 745 (2006).

18. Hamdy M. Youssef and Amnah H. Al-Harby, J. Archive Appl. Mech. 77, 675 (2007). doi:10.1007/s00419-007-0120-6.
19. Hamdy M. Youssef, Comp. Math. Modeling 19, 201 (2008). doi:10.1007/s10598-008-0014-7.
20. Magdy A. Ezzat and Hamdy M. Youssef, J. Therm. Stresses 32, 414 (2009). doi:10.1080/01495730802637233.
21. Hamdy M. Youssef and A.A. El-Bary, J. Therm. Stresses 32, 1 (2009). doi:10.1080/01495730903249276.
22. Magdy A. Ezzat, Can. J. Phys. 86, 1241 (2008). doi:10.1139/P08-069.
23. L. Y. Bahar and R. B. Hetnarski, J. Therm. Stresses 1, 135 (1978). doi:10.1080/01495737808926936.
24. L. Y. Bahar and R. B. Hetnarski, State space approach to thermoelasticity, in Proceedings of 6th Canadian Congress of Applied Mechanics, University of British Columbia, Vancouver, British Columbia, Canada, pp. 17-18, 1977.
25. L. Y. Bahar and R. B. Hetnarski, Transfer matrix approach to thermoelasticity, in Proceedings of 15th Midwest Mechanical Conference, University of Illinois at Chicago Circle, pp. 161-163, 1977.
26. D. Tzou, Macro-to-micro heat transfer, Taylor & Francis, Washington DC, 1996.

Appendix

$L_{11}(x,s)= a_0 - a_4\alpha_3,$

$L_{12}(x,s) =- a_4\alpha_4,$

$L_{13}(x,s) = a_2 + a_4\alpha_5,$

$L_{14}(x,s) = a_1 - a_5\alpha_3,$

$L_{15}(x,s) = - a_5\alpha_4,$

$L_{16}(x,s)=a_3 + a_5\alpha_5,$

$L_{21}(x,s) = a_4\alpha_2\alpha_3,$

$L_{22}(x,s) = a_0 + a_2\alpha_1 + a_4\left(\alpha_1^2 +\alpha_2\alpha_4\right),$

$L_{23}(x,s) =- a_2\alpha_2 - a_4\alpha_2\left(\alpha_1 + \alpha_5\right),$

$L_{24}(x,s) = a_5\alpha_2\alpha_3,$

$L_{25}(x,s)= a_1 + a_3\alpha_1 + a_5(\alpha_1^2 + \alpha_2\alpha_4),$

$L_{26}(x,s)=- a_3\alpha_2 - a_5\alpha_2(\alpha_1 + \alpha_5),$

$L_{31}(x,s)=- a_2\alpha_3 - a_4\alpha_3\alpha_5,$

$L_{32}(x,s)=- a_2\alpha_4 - a_4\alpha_4(\alpha_1 + \alpha_5),$

$L_{33}(x,s)= a_0 + a_2\alpha_5 + a_4(\alpha_2\alpha_4 - \alpha_3 + \alpha_5^2),$

$L_{34}(x,s)=- a_3\alpha_3 - a_5\alpha_3\alpha_5,$

$L_{35}(x,s)= - a_3\alpha_4 - a_5\alpha_4(\alpha_1 + \alpha_5),$

$L_{36}(x,s)= a_1 + a_3\alpha_5 + a_5(\alpha_2\alpha_4 - \alpha_3 + \alpha_5^2),$

$L_{41}(x,s)= - a_3\alpha_3 - a_5\alpha_3\alpha_5,$

$L_{42}(x,s)= - a_3\alpha_4 - a_5\alpha_4(\alpha_1 + \alpha_5),$

$L_{43}(x,s)= a_1 + a_3\alpha_5 + a_5(\alpha_2\alpha_4 - \alpha_3 + \alpha_5^2),$

$L_{44}(x,s)= a_0 - a_4\alpha_3,$

$L_{45}(x,s)=- a_4\alpha_4,$

$L_{46}(x,s)= a_2 + a_4\alpha_5,$

$L_{51}(x,s)= a_3\alpha_2\alpha_3 + a_5\alpha_2\alpha_3(\alpha_1 + \alpha_5),$

$$L_{52}(x,s)= a_1\alpha_1 + a_3(\alpha_1^2 + \alpha_2\alpha_4) + a_5(\alpha_1^3 + 2\alpha_1\alpha_2\alpha_4 + \alpha_2\alpha_4\alpha_5),$$

$$L_{53}(x,s)=- a_1\alpha_2 - a_3\alpha_2(\alpha_1 + \alpha_5) - a_5\alpha_2(\alpha_1^2 + \alpha_1\alpha_5 + \alpha_2\alpha_4 - \alpha_3 + \alpha_5^2),$$

$L_{54}(x,s)= a_4\alpha_2\alpha_3,$

$L_{56}(x,s)=- a_2\alpha_2 - a_4\alpha_2(\alpha_1 + \alpha_5),$

$L_{61}(x,s)=- a_1\alpha_3 - a_3\alpha_3\alpha_5 - a_5\alpha_3(\alpha_2\alpha_4 - \alpha_3 + \alpha_5^2),$

$$L_{62}(x,s)=- a_1\alpha_4 - a_3\alpha_4(\alpha_1 + \alpha_5) - a_5\alpha_4(\alpha_1^2 + \alpha_1\alpha_5 + \alpha_2\alpha_4 - \alpha_3 + \alpha_5^2),$$

$$L_{63}(x,s)=a_1\alpha_5 + a_3(\alpha_2\alpha_4 - \alpha_3 + \alpha_5^2) + a_5(\alpha_1\alpha_2\alpha_4+\alpha_5(2\alpha_2\alpha_4-2\alpha_3+\alpha_5^2)),$$

$L_{64}(x,s)= - a_2\alpha_3 - a_4\alpha_3\alpha_5,$

$L_{65}(x,s)=- a_2\alpha_4 - a_4\alpha_4(\alpha_1 + \alpha_5),$

$L_{66}(x,s)=a_0 + a_2\alpha_5 + a_4(\alpha_2\alpha_4 - \alpha_3 + \alpha_5^2),$